DPS Day 2023

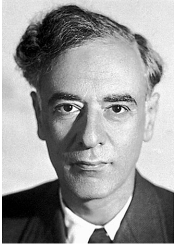
IISER Kolkata

17-18th March 2023

Graph Polynomials & Topological Entanglement Entropy

Entropy

(1932) Introduction of math framework for quantum mechanics by John von Neumann



(1927) Lev Landau & Neumann introduced density matrices to study quantum states of a physical system

If ρ = density matrix, then $S(\rho) = -\text{trace}(\rho \ln \rho)$ is the von Neumann entropy.

ENTANGLEMENT & ORDER

(von Neumann) entropy is a quantitive measure of entanglement in a quantum system. It measures local and non-local quantum correlations.

There is a notion of topological order in quantum matter. Microscopically, these correspond to long-range quantum entanglement.

The ground state of a quantum-mechanical system is its stationary state of lowest energy.

Topological Entanglement Entropy

Kitaev and Preskill (2006) showed that if we consider a planar disk A of radius $L/2\pi$, L being sufficiently large, then in the ground state we have

$$S(\rho) = \alpha L - \gamma + \cdots$$

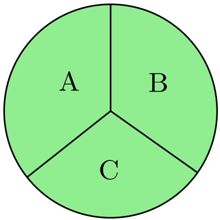
- The ellipsis is a sum of terms that vanish as $L \to \infty$.
- The term γ is *geometry independent*.
- γ captures global features of entanglement in ground state.

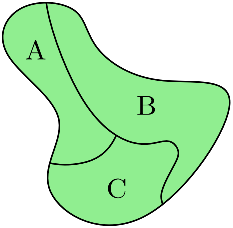
We define $-\gamma$ to be the topological entanglement entropy.

TEE: Example I

We shall often refer to $-\gamma$ as either TEE or S_{topo}.

We have a planar CSS (collection of subsystems) with 3 regions.



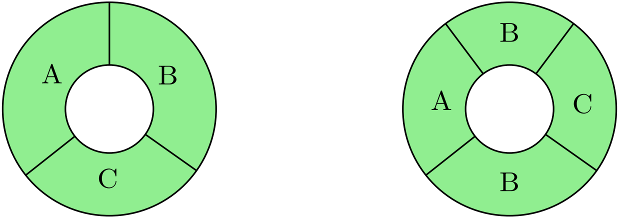


Note that the tripartite information

$$I_3(A,B,C) := S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC} = -\gamma.$$

TEE: Example II, III

Consider the CSS with 3 regions arranged in an annulus.



Let $I_3(A,B,C) = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC}$.

In both cases, $I_3 = -2\gamma = 2S_{\text{topo}}$.

The factor of 2 *is* the Euler characteristic χ (of S^2)!

A Generalization

Question How to generalize the tripartite information I_3 to a CSS with N subsystems such that

- the CSS may lie on a (non-planar) surface
- works for all arrangements of CSS
- and captures S_{topo}?

Assumption (i) Any two subsystem in a given CSS may not overlap but can intersect only along its boundary.

(ii) Three of more subsystems do not intersect.

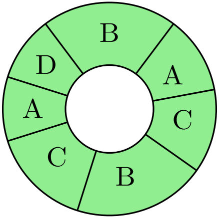
Answer For planar annular arrangement of CSS (say $N = 4$),

$$\begin{aligned} I_4(A,B,C,D) &= (S_A + S_B + S_C + S_D) - (S_{AB} + \cdots + S_{CD}) \\ &\quad + (S_{ABC} + \cdots + S_{BCD}) - S_{ABCD}. \end{aligned}$$

Multipartite captures TEE

Theorem [Patra, -, Lal] *We have $|I_N| = \chi S_{\text{topo}}$ for any annular arrangement.*

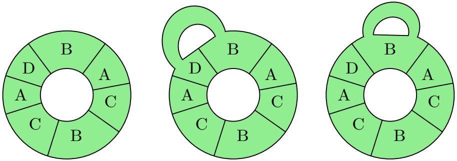
The annular arrangement could be as in Example II as well as III, i.e., if we have $A_1, \ldots, A_N$ subsystems forming the CSS, then the arrangement of the totality of A_i's should look like an annulus. However, each A_i may have many components.



Properties

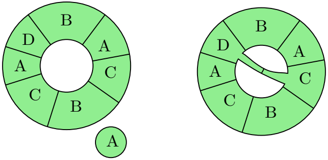
(A) Multipartite information is preserved under

- attaching self loops
- attaching nearest neighbour handles

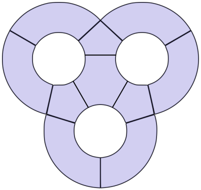


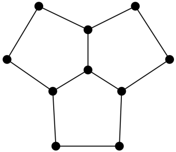
(B) Multipartite information is annihilated under

- adding a disjoint subsystem
- adding a short-circuit



From CSS to Graphs & Back





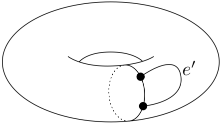
Warning You have to assign a vertex for each component of A_i. But now vertices assigned to one subsystem A_i will have the same colour!

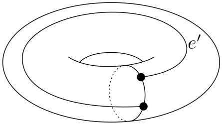
How do we go from a graph to CSS? **Thicken!**

Graph Polynomial

Theorem [Bhasin, -, Patra, Lal] *For any vertex coloured graph* Γ *embedded in a surface, there is a polynomial* $\beta_\Gamma(x)$ *such that the multipartite information is* $\beta_\Gamma(-1)$.

A fixed graph may have many non-isomorphic embeddings in a fixed surface. The polynomial may also be different, thereby distinguishing the embeddings.





References

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S. Patra, S. Basu, S. Lal
Phys. Rev. A, **105**, 052428 [2022]

Graph Polynomials for Coloured Embedded Graphs: A Topological Approach
D. Bhasin, S. Basu, S. Patra, S. Lal
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